

MODIFIED KERNELS IN SPECTRAL GEOID DETERMINATION: FIRST RESULTS FROM WESTERN AUSTRALIA

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Abstract

The deterministic kernel modifications proposed by Wong and Gore (1969), Meissl (1971), and Vanicek and Kleusberg (1987) have been implemented in the one-dimensional fast Fourier transform (1D-FFT) approach to geoid determination. Comparisons with discrete geoid heights provided by Global Positioning System and Australian Height Datum data in Western Australia illustrate consistent improvements when using spherical integration caps in conjunction with a modified kernel.

Introduction

In the mid 1980s, the fast Fourier transform (FFT) began to find wide-spread use in geoid determination because of its efficient evaluation of convolution integrals, when compared to classical numerical integration. For many years, the planar, two-dimensional FFT was used (eg. Schwarz *et al.* 1990). Strang van Hees (1990) then introduced the spherical, two-dimensional FFT. However, both of these FFT approaches are subject to approximation errors, the most notable of which is the simplification of Stokes's kernel. Therefore, Forsberg and Sideris (1993) proposed the spherical, multi-band FFT, which reduces the impact of the simplified kernel. Haagmans *et al.* (1993) then refined this approach to give the spherical, one-dimensional FFT, which requires no simplification of Stokes's kernel. Ironically, the 1D-FFT approach is a combination of the FFT and numerical integration, but is indisputably faster than numerical integration alone.

Regional gravimetric determinations of the geoid using the FFT often convolve the whole rectangular grid of gravity anomalies with the Stokes kernel. Conversely, geoid determinations using numerical integration convolve residual gravity over a spherical cap of constant radius (ψ_0) about each computation point with Stokes's kernel. Whether the integration is performed over a spherical cap or the whole data area, this still represents an approximation of Stokes's integral. Each approach results in a truncation error due to the neglect of the residual gravity anomalies in the remote zones outside the integration domain. This truncation error is commonly neglected during practical geoid computations, particularly when using the fast Fourier transform approach (eg. Schwarz *et al.* 1990). However, such an assumption is not strictly valid because neither the integration kernel nor the residual gravity anomalies are necessarily zero outside the integration domain.

Instead, the impact of this truncation error should be reduced through a modification to Stokes's integration kernel. Several authors have proposed reductions of the truncation error in numerical integration over a spherical cap. These include deterministic kernel modifications (eg. Molodensky *et al.* 1962, Wong and Gore 1969, Meissl 1971, Vanicek and Kleusberg 1987) and stochastic modifications (eg. Wenzel 1982, Sjoberg 1991, Vanicek and Sjoberg 1991). Kernel modification methods have also been proposed to reduce the truncation error for integration areas other than a spherical cap (eg. Neyman *et al.* 1996, Zelin and Zoufa 1992).

The deterministic kernel modifications

Wong and Gore (1969)

Wong and Gore (1969) and Vanicek and Kleusberg (1987) propose to remove the low-degree Legendre polynomials from Stokes's kernel, which also reduces the magnitude of the truncation error. This will be called the Wong and Gore kernel, and is given by

$$S^M(\Delta\lambda) = S(\Delta\lambda) - \sum_{n=2}^M \frac{2n+1}{n-1} P_n(\Delta\lambda) \quad \text{for } 0 \leq \Delta\lambda \leq \psi_0, \quad (1)$$

where $P(\Delta\lambda)$ are Legendre's polynomials along the parallel as used in the 1D-FFT, and M is the degree of spheroidal modification to the integration kernel.

If the Stokesian convolution is performed over the whole sphere using the Wong and Gore kernel, the orthogonality relations dictate that the result is identical to that achieved when the low frequencies ($2 \leq n \leq M$) are removed from the kernel, the gravity anomalies, or both. However, when the integration is performed over a limited area, as is the case in any regional geoid determination, the Wong and Gore kernel only acts as a partial filter. Therefore, if the degrees of kernel modification (M) and geopotential model (L) are different, different results can be expected.

Meissl (1971)

Meissl's (1971) kernel modification simply subtracts the value of Stokes's kernel at the truncation radius from the kernel. Thus, the Meissl kernel is

$$S_{me}(\psi) = S(\psi) - S(\psi_0) \quad \text{for } 0 \leq \Delta\lambda \leq \psi_0. \quad (2)$$

This modification causes the Fourier series of the truncation error to converge to zero more rapidly, and can be proven by applying Green's second identity (eg. Jekeli 1981).

Vanicek and Kleusberg (1987)

Vanicek and Kleusberg (1987) and Vanicek and Sjöberg (1991) apply theory similar to that used by Molodensky *et al.* (1962) to Eq. (1). This modification gives

$$S^{M*}(\Delta\lambda) = S^M(\Delta\lambda) - \sum_{n=2}^M \frac{2n+1}{2} t_n P_n(\Delta\lambda) \quad \text{for } 0 \leq \Delta\lambda \leq \psi_0, \quad (3)$$

where the coefficients t_n in Eq. (3) are analogous with the spherical truncation coefficients Q_n defined by Molodensky *et al.* (*ibid.*), and are given by Vanicek and Sjöberg (*ibid.*) as:

$$\sum_{n=2}^M \frac{2n+1}{2} t_n e_{kn} = Q_k - \sum_{n=2}^M \frac{2n+1}{n-1} e_{kn}, \quad (4)$$

where the coefficients e_{kn} can be computed using the recursive relations of Paul (1973).

Comparison of Modified 1D-FFT Results with GPS at AHD Benchmarks

In this investigation, the deterministic kernel modifications in Eqs. (1), (2) and (3) were implemented in the 1D-FFT computer software developed at the University of Calgary (Sideris 1994). The stochastic kernel modifications were not considered because accurate estimates of the errors in the terrestrial gravity data are not currently available. In order to prevent the whole grid of gravity data being used during the 1D-FFT geoid computation, the kernel was set to zero outside the cap radius (ψ_0) before being transformed to the frequency domain. Similarly, each modified kernel was computed before its transformation to the frequency domain. As is customary, the 1D-FFT geoid results were compared with Global Positioning System (GPS) and optical levelling data to determine if any improvements are made when utilising integration caps and modified kernels.

All geoid computations were conducted on a 5' by 5' grid over an area bound by $37^\circ\text{S} < \phi < 13^\circ\text{S}$ and $112^\circ\text{E} < \lambda < 131^\circ\text{E}$, which completely covers Western Australia. The terrestrial gravity data used comprise simple 5' by 5' means of the validated Australian gravity database (Featherstone *et al.* 1997), supplemented with marine gravity anomalies derived from combined satellite altimeter missions (Sandwell *et al.* 1995). These free-air gravity data were reduced by the $L=360$ expansion of EGM96 (Lemoine *et al.* 1997), then gridded using splines in tension (Smith and Wessel 1990). Digital elevation models were not used in the computations because the direct and indirect effects of the terrain are known to be small over most of Western Australia (Zhang and Featherstone 1997). The 1D-FFT geoid solutions, computed using various cap-radii and modified kernels, were interpolated and compared with 65 discrete geoid heights from Western Australian GPS networks (Morgan *et al.* 1996, Stewart *et al.* 1997) co-located with optically levelled heights on the Australian Height Datum (AHD). The results are shown in Table 1.

Table 1. Statistical fit of the modified 1D-FFT free-air co-geoid models to 65 discrete GPS-AHD geoid heights in Western Australia (units in metres).

kernel	degree	ψ_0 (°)	max.	min.	mean	std	rms
EGM96 only	$L=360$	n/a	1.831	-0.255	1.024	0.345	1.080
Stokes	n/a	0.25	1.781	-0.831	1.054	0.376	1.119
“	“	0.50	1.711	-0.660	1.051	0.372	1.115
“	“	1.00	1.904	-0.632	1.044	0.411	1.122
“	“	1.50	2.127	-0.669	1.047	0.440	1.136
“	“	2.00	2.251	-0.564	1.045	0.459	1.141
“	“	30.00	2.291	-0.438	1.179	0.491	1.276
Wong and Gore	$M=360$	0.25	1.823	-0.533	1.032	0.343	1.087
“	“	0.50	1.851	-0.603	1.034	0.358	1.094
“	“	1.00	1.834	-0.626	1.035	0.362	1.097
“	“	1.50	1.826	-0.636	1.035	0.363	1.097
“	“	2.00	1.821	-0.629	1.036	0.363	1.098
“	“	30.00	1.823	-0.631	1.036	0.364	1.100
Wong and Gore	$M=36$	0.25	1.786	-0.796	1.051	0.371	1.114
“	“	0.50	1.733	-0.669	1.049	0.364	1.111
“	“	1.00	1.709	-0.643	1.045	0.374	1.110
“	“	1.50	1.770	-0.660	1.045	0.379	1.112
“	“	2.00	1.768	-0.658	1.045	0.379	1.111
“	“	30.00	1.771	-0.611	1.073	0.387	1.140
Meissl	n/a	0.25	1.811	-0.624	1.039	0.350	1.096
“	“	0.50	1.777	-0.685	1.046	0.375	1.105
“	“	1.00	1.732	-0.648	1.045	0.362	1.106
“	“	1.50	1.746	-0.659	1.045	0.377	1.111
“	“	2.00	1.849	-0.646	1.044	0.389	1.114
“	“	30.00	2.266	-0.464	1.153	0.492	1.125
Vanicek/Kleusberg	$M=36$	0.25	1.786	-0.798	1.051	0.371	1.115
“	“	0.50	1.730	-0.668	1.050	0.365	1.111
“	“	1.00	1.752	-0.641	1.045	0.381	1.112
“	“	1.50	1.873	-0.663	1.046	0.394	1.118
“	“	2.00	1.933	-0.627	1.044	0.400	1.119
“	“	30.00	2.146	-0.537	1.101	0.456	1.192

In Table 1, a ~1m bias exists between the results because the zero-degree term in the gravimetric geoid has been considered (Kirby and Featherstone 1997), and there is a ~0.7m separation between the W_0 =constant geoid and the AHD (Rapp 1994). It is most interesting to note that the inclusion of the Australian and satellite altimeter gravity data often degrades the standard deviations of the agreements with the 65 control points. This effect becomes more pronounced with increasing cap-radius, and is most probably due to a combination of:

errors in the satellite altimeter and Australian gravity data, improper gridding using only free-air anomalies, the neglect of terrain effects, and errors in the control data through the GPS observations or distortions in the AHD in Western Australia. Nevertheless, as this investigation is only concerned with the relative merits of using caps and modified kernels in the 1D-FFT, the exact origin or removal of these errors becomes immaterial.

The computations utilised cap-radii ranging from 0.25° (the Nyquist frequency of EGM96) to 2° (the minimum radius from the edge of the data area and a control point). A cap-size of 30° has also been specified so as to allow the whole gravity data grid to be used by way of comparison. The most striking observation is that the use of a cap in the 1D-FFT improves the standard deviation of the comparisons over those achieved when using the whole data area (eg. 0.372m for $\psi_0=0.5^\circ$ versus 0.491m for the whole grid). This clearly illustrates that it is preferable to implement the 1D-FFT with a limited spherical cap.

These improved results are achieved because the limited spherical cap filters out some of the effects of the errors in the Australian gravity data in remote areas, where they start to have a detrimental effect on the geoid solution. This observation also applies to the three deterministically modified forms of Stokes's kernel. Moreover, the modified kernels yield improved agreements over the unmodified Stokes kernel for each cap-radius. This indicates that it is preferable to use a limited spherical cap and a modified integration kernel in 1D-FFT geoid computations. Indeed, this approach makes the spectral and spatial approaches to geoid computation even more similar, but with the attraction of computational efficiency offered by the Fourier-based methods.

Lastly, the $M=36$ and $M=360$ spheroidal modifications have been used for the Wong and Gore kernel in order to illustrate the (partial) high-pass filtering effect of a high degree of spheroidal modification. The $M=360$ modification gives slightly better agreements than the $M=36$ kernel because it filters out most of the medium frequency errors from the Australian gravity field. Note also that the statistics of the differences in Table 1 are weakly dependent on the cap-radius for the $M=360$ modification. This is because the modification is a more efficient high-pass filter than a limited spherical cap.

Conclusions

From these preliminary geoid results over Western Australia, it is clear that spectral geoid computations should use a spherical cap of limited extent instead of the whole gravity data grid. This causes the FFT approach to more closely mimic numerical integration. The use of a limited spherical cap also acts as a (partial) high-pass filter, and thus removes the detrimental effects of erroneous gravity data. Moreover, small but consistent improvements are observed when a deterministically modified integration kernel is also used. This is because the kernel modification reduces the truncation error associated with performing the convolution over a limited area. Therefore, it is recommended that limited integration caps and modified kernels are used in future spectral geoid determinations.

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